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ASYMPTOTIC BEHAVIOUR OF REPRODUCING KERNELS OF WEIGHTED BERGMAN SPACES

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ABSTRACT. Let Ω be a domain in $\mathbb{C}^n$, F a nonnegative and G a positive function on Ω such that $1/G$ is locally bounded, A^2_α the space of all holomorphic functions on Ω square-integrable with respect to the measure $F^\alpha G\,d\lambda$, where $d\lambda$ is the $2n$-dimensional Lebesgue measure, and $K_\alpha(x,y)$ the reproducing kernel for A^2_α. It has been known for a long time that in some special situations (such as on bounded symmetric domains Ω with $G=\mathbf{1}$ and $F=$ the Bergman kernel function) the formula

$$(*)\qquad \lim_{\alpha\to+\infty} K_\alpha(x,x)^{1/\alpha} = 1/F(x)$$

holds true. [This fact even plays a crucial role in Berezin's theory of quantization on curved phase spaces.] In this paper we discuss the validity of this formula in the general case. The answer turns out to depend on, loosely speaking, how well the function $-\log F$ can be approximated by certain pluriharmonic functions lying below it. For instance, $(*)$ holds if $-\log F$ is convex (and, hence, can be approximated from below by linear functions), for any function G. Counterexamples are also given to show that in general $(*)$ may fail drastically, or even be true for some x and fail for the remaining ones. Finally, we also consider the question of convergence of $K_\alpha(x,y)^{1/\alpha}$ for $x\neq y$, which leads to an unexpected result showing that the zeroes of the reproducing kernels are affected by the smoothness of F: for instance, if F is not real-analytic at some point, then $K_\alpha(x,y)$ must have zeroes for all α sufficiently large.

1. Introduction and results

Let Ω be a domain in $\mathbb{C}^n$ and F,G nonnegative measurable functions on Ω such that $G>0$ and $1/G$ is locally bounded. The weighted Bergman space appearing in the title is

$$A^2_\alpha = \{f \text{ holomorphic on } \Omega:\ \left(\textstyle\int_\Omega |f|^2 F^\alpha G\,d\lambda\right)^{1/2} \equiv \|f\|_\alpha < +\infty\}.$$

Here $d\lambda$ stands for the Lebesgue measure and α is a real number.

The reproducing kernel for A^2_α is the function $K_\alpha(x,y)$ of two variables $x,y\in\Omega$, holomorphic in x and anti-holomorphic in y, such that $K_\alpha(\,\cdot\,,y)\in A^2_\alpha$ for each y and

$$f(y) = \int_\Omega f(x)\overline{K_\alpha(x,y)}F(x)^\alpha G(x)\,d\lambda(x) \qquad \forall y\in\Omega\ \forall f\in A^2_\alpha.$$

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Under suitable hypothesis on F (for instance, when $F > 0$ and $1/F$ is locally bounded) and the stated hypothesis on G, it is known that the reproducing kernel exists and is unique ([6], [15]) and the value $K_\alpha(x,x)$ coincides with the square $e_\alpha(x)$ of the norm of the evaluation functional at x on A^2_α:

$$K_\alpha(x,x) = e_\alpha(x) \equiv \sup\{|f(x)|^2;\ f \in A^2_\alpha, \|f\|_\alpha \le 1\}.$$

Our main concern here will be the limit

$$\rho(x) \equiv \lim_{\alpha\to+\infty} K_\alpha(x,x)^{1/\alpha} = \lim_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha}.$$

There is *a priori* no reason for this limit even to exist. However, in many important situations the limit does exist, and, moreover, is equal to

$$\rho(x) = \frac{1}{F(x)}. \tag{1}$$

Instances of this situation include the following:

1. $\Omega = \mathbb{D}$, the unit disc in $\mathbb{C}$; $G = \mathbf{1}$ (the constant one), $F(z) = 1 - |z|^2$. (Folklore; known, at least, already to Poincaré.)
2. The Segal-Bargmann (or Fock) spaces: $\Omega = \mathbb{C}^n$, $G = \mathbf{1}$, $F = e^{-|z|^2}$ ([4], [5], [17], [1]).
3. (a generalization of 1.) $\Omega =$ the unit ball of $\mathbb{C}^n$, $G = \mathbf{1}$, $F = 1 - \|z\|^2$ [22].
4. $\Omega =$ a bounded symmetric domain, $G = \mathbf{1}$, $F =$ the Bergman kernel function ([3], [16], [20], [12], [13]).
5. Ω a domain in $\mathbb{C}$ of hyperbolic type, $G = \mathbf{1}$, $F(\phi(z)) = (1 - |z|^2) \cdot |\phi'(z)|$ where $\phi : \mathbb{D} \to \Omega$ is any uniformization map (that is, $ds^2 = F(z)\,|dz|^2$ is the Poincaré metric on Ω) ([19], [9]).
6. Some pseudoconvex domains in $\mathbb{C}^2$ equipped with a Kähler metric $g_{i\bar{j}} dz_i d\bar{z}_j$, with $g_{i\bar{j}} = (\partial^2\Psi/\partial z_i \partial\bar{z}_j)$, $F = e^{-\Psi}$, $G = \det(g_{i\bar{j}})$, where Ψ is a real-valued strictly plurisubharmonic function (the Kähler potential) [11].

In this note we will show that the following general result holds.

Theorem A. *Let $F \ge 0$ and $G > 0$ be measurable functions on Ω such that $1/G$ is locally bounded. Suppose that $-\log F$ is a convex function and that $\mathbf{1} \in A^2_\alpha$ for some $\alpha > 0$. Then the limit*

$$\lim_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} \equiv \rho(x)$$

exists and is equal to $1/F(x)$.

The limits above are of central importance in some approaches to quantization on Ω. See [3], [10], [11], [8], [26], [25].

A more detailed description involves the auxiliary functions $F^*, F^{**}, F^{***}, F^\#$ defined by

$$1/F^* = \sup\{|e^g|^2 :\ g \text{ holomorphic on } \Omega, |e^g|^2 \le 1/F, e^{\alpha g} \in A^2_\alpha \text{ for some } \alpha > 0\},$$
$$1/F^{**} = \sup\{|e^g|^2 :\ g \text{ holomorphic on } \Omega, |e^g|^2 \le 1/F\},$$
$$1/F^{***} = \sup\{|f|^\varkappa :\ f \text{ holomorphic on } \Omega, \varkappa > 0, |f|^\varkappa \le 1/F\},$$
$$1/F^\# = \sup\{e^\psi :\ \psi \text{ is plurisubharmonic and } e^{\psi(x)} \le 1/\lim_{\epsilon\searrow 0} \inf_{|x-y|<\epsilon} F(y)\}.$$

(In other words, for F lower semicontinuous, $-\log F^\#$ is the greatest plurisubharmonic function majorized by $-\log F$.) The condition that $e^{\alpha g} \in A^2_\alpha$ for some $\alpha > 0$

can be equivalently stated as $|e^g|^2 F \in L^\alpha(\Omega, G\,d\lambda)$ for some $\alpha > 0$; if the measure $G\,d\lambda$ is finite, this condition can even be omitted completely, and $F^* = F^{**}$. In general, we only have $F^* \geq F^{**} \geq F^{***} \geq F$ and $F^{***} \geq F^{\#} \geq F_{\rm lsc}$, where $F_{\rm lsc}$ denotes the lower-semicontinuous regularization of F.

Theorem B. *Let $F \geq 0$ and $G > 0$ be measurable functions on Ω such that $1/G$ is locally bounded. Then*

$$1/F^{\#}(x) \geq \limsup_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} \geq \liminf_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} \geq 1/F^*(x).$$

If $\mathbf{1} \in A^2_\alpha$ for some $\alpha > 0$, then even

$$\liminf_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} \geq 1/F^{**}(x).$$

We also have a sharper lower bound for $\limsup_{\alpha\to+\infty} e_\alpha^{1/\alpha}$:

Theorem B′. *Let $F \geq 0$ and $G > 0$ be measurable functions on Ω such that $1/G$ is locally bounded and $\mathbf{1} \in A^2_{\alpha_0}$ for some $\alpha_0 > 0$. Then*

$$\limsup_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} \geq 1/F^{***}(x).$$

If F is also positive and $1/F$ locally bounded (so that the reproducing kernels $K_\alpha(x,y)$ — not only $e_\alpha(x)$ — are defined), we can also consider the convergence of $K_\alpha(x,y)^{1/\alpha}$ on $\Omega\times\Omega$, provided the α-th root makes sense. This will reveal the following surprising connection between the zeroes of $K_\alpha(x,y)$ and the smoothness of F.

Theorem C. *Assume that $F, G > 0$, $1/F$ and $1/G$ are locally bounded and the limit $\rho(x)$ exists and equals $1/F$. Suppose further that there exist an unbounded subset A of $[1,+\infty)$ and a simply-connected open set $U \subset \Omega$ such that*

$$K_\alpha(x,y) \neq 0 \qquad \text{for all } \alpha \in A \text{ and } x, y \in U.$$

Then $F(x)$ extends to a zero-free function $F(x,y)$ on $U \times U$, holomorphic in x and anti-holomorphic in y, such that $F(x,x) = F(x)$ and $|F(x,y)|^2 \geq F(x,x)F(y,y)$.

Corollary. *Assume that $F, G > 0$, $1/F$ and $1/G$ are locally bounded and the limit $\rho(x)$ exists and equals $1/F$. Suppose further that F is not real analytic at some point $z_0 \in \Omega$. Then for any sequence $\alpha_k \to \infty$ there exist a subsequence α_{k_j} and points $x_j, y_j \in \Omega$ such that both $\{x_j\}$ and $\{y_j\}$ converge to z_0 and $K_{\alpha_{k_j}}(x_j, y_j) = 0$ for each j. (In other words, the point (z_0, z_0) is an accumulation point of zeroes of the functions $K_\alpha(x,y)$.)*

Theorems A, B and B′ are proved in Section 2, Theorem C in Section 3. Section 4 brings some examples, and the last Section 5 mentions briefly some open problems.

Throughout the text, the letters $d\mu$ stand for the measure $d\mu(x) = G(x)\,d\lambda(x)$, and $d\mu_\alpha$ denotes the measure $d\mu_\alpha = F^\alpha d\mu$. If a function u taking values in the interval $[-\infty, +\infty)$ is locally bounded from above, we will denote by $u_{\rm usc}$ its upper-semicontinuous regularization

$$u_{\rm usc}(x) := \lim_{\epsilon\searrow 0} \sup_{y\in D(x,\epsilon)} u(y).$$

Clearly $u \leq u_{\text{usc}}$, and equality prevails iff u is upper semicontinuous. The lower-semicontinuous regularization can be defined analogously. The functions $\overline{\rho}(x)$ and $\underline{\rho}(x)$ are abbreviations for

$$\overline{\rho}(x) := \limsup_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} \quad \text{and} \quad \underline{\rho}(x) := \liminf_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha},$$

respectively; and PSH stands for "plurisubharmonic".

2. The limit of $e_\alpha(x)^{1/\alpha}$

Proof of Theorem B. Let $r > 0$ be such that the closed polydisc $\overline{D} = \overline{D(x,r)}$ lies wholly in Ω. By the mean value theorem for holomorphic functions,

$$f(x) = (\pi r^2)^{-n} \int_D f\, d\lambda = (\pi r^2)^{-n} \int_D \frac{f}{F^\alpha G}\, d\mu_\alpha$$

for any holomorphic function f on Ω. By the Schwarz inequality,

$$\begin{aligned} |f(x)|^2 &\leq \|f\|_\alpha^2 \cdot \Big(\int_D (\pi r^2)^{-2n} F^{-2\alpha} G^{-2}\, d\mu_\alpha \Big) \\ &= \|f\|_\alpha^2 \cdot (\pi r^2)^{-2n} \int_D F^{-\alpha} G^{-1}\, d\lambda. \end{aligned}$$

It follows that

$$\begin{aligned} e_\alpha(x) &\leq (\pi r^2)^{-2n} \int_D F^{-\alpha} G^{-1}\, d\lambda \\ &\leq (\pi r^2)^{-n} \cdot \sup_D \frac{1}{G} \cdot (\inf_D F)^{-\alpha}. \end{aligned}$$

Taking roots gives

$$e_\alpha(x)^{1/\alpha} \leq \left(\frac{\sup_D 1/G}{\pi^n r^{2n}} \right)^{1/\alpha} \cdot (\inf_D F)^{-1}. \tag{2}$$

Note that the last supremum is finite by hypothesis. Letting α tend to infinity, we therefore obtain

$$\overline{\rho}(x) \leq 1/\inf_{D(x,r)} F.$$

This holds for all sufficiently small positive r. Letting $r \to 0$ yields

$$\overline{\rho}(x) \leq 1/\lim_{r\searrow 0} \inf_{D(x,r)} F = 1/F_{\text{lsc}}(x).$$

On the other hand, by the definition of e_α,

$$e_\alpha(x)^{1/\alpha} = \sup\{|f(x)|^{2/\alpha} : \ f \text{ holomorphic on } \Omega, \|f\|_\alpha \leq 1\}.$$

Consequently,

$$\log e_\alpha(x)^{1/\alpha} = \sup\{\frac{2}{\alpha} \log|f(x)| : \ f \text{ holomorphic on } \Omega, \|f\|_\alpha \leq 1\},$$

and

$$\log \overline{\rho}(x) = \lim_{k\to+\infty} \sup\{\frac{2}{\alpha} \log|f(x)| : \ \alpha \geq k, \|f\|_\alpha \leq 1, f \text{ holomorphic on } \Omega\}. \tag{3}$$

Recall now the following well-known facts from the theory of plurisubharmonic functions:

(a) *If u_k is a decreasing sequence of PSH functions, then $u = \lim_{k\to\infty} u_k$ is also plurisubharmonic.*

(b) *If* $\{u_\iota\}_{\iota\in I}$ *is a family of PSH functions such that its supremum* $u=\sup_{\iota\in I}u_\iota$ *is locally bounded from above, then the upper-semicontinuous regularization* $u_{\rm usc}$ *of* u *is also plurisubharmonic.*

(c) *Let* u *be a function locally bounded from above and* $u_r(x):=\sup_{D(x,r)}u$ *(so* $u_{\rm usc}=\lim_{r\searrow 0}u_r$*). Then* $\lim_{r\searrow 0}(u_r)_{\rm usc}=u_{\rm usc}$*.*

[For proofs of (a) and (b) see e.g. [18], Theorem 2.9.14. For (c), observe first that $(u_r)_{\rm usc}\le u_{(1+\delta)r}$ for any $\delta>0$, by the triangle inequality; combining this with the trivial fact that $(u_r)_{\rm usc}\ge u_r$ and letting r tend to zero gives the result.]

For brevity, let us temporarily denote

$$\begin{aligned} U_k(x) &:= \sup\{\frac{2}{\alpha}\log|f(x)|:\ \alpha\ge k, \|f\|_\alpha\le 1, f \text{ holomorphic on } \Omega\} \\ &= \sup_{\alpha\ge k}\log e_\alpha(x)^{1/\alpha}, \\ C_r(x) &:= \max\left[1,\frac{\sup_{D(x,r)}1/G}{\pi^n r^{2n}}\right]. \end{aligned}$$

Then $U_k\searrow\log\overline{\rho}$ as $k\to\infty$ and, in view of (2),

$$U_k(x)\le\sup_{\alpha\ge k}\log\frac{C_r(x)^{1/\alpha}}{\inf_{D(x,r)}F}=\log\frac{C_r(x)^{1/k}}{\inf_{D(x,r)}F},$$

so the functions U_k are locally bounded from above. Moreover,

$$\begin{aligned} \lim_{k\to\infty}(U_k)_{\rm usc} &\le \lim_{k\to\infty}\left[\frac{1}{k}(\log C_r)_{\rm usc}+(-\log\inf_{D(x,r)}F)_{\rm usc}\right] \\ &= (\log\sup_{D(x,r)}1/F)_{\rm usc}. \end{aligned}$$

The left-hand side is independent of r; letting $r\to 0$ yields, by (c) above,

$$\lim_{k\to\infty}(U_k)_{\rm usc}\le(-\log F)_{\rm usc}.$$

In view of (b), and since $\log|f|$ is plurisubharmonic for any holomorphic function f, each $(U_k)_{\rm usc}$ is a PSH function. The sequence U_k being decreasing, (a) implies that $\lim_{k\to\infty}(U_k)_{\rm usc}$ is also plurisubharmonic. Since the greatest PSH function majorized by $(-\log F)_{\rm usc}$ is $-\log F^{\#}$ by definition, we see that

$$\lim_{k\to\infty}(U_k)_{\rm usc}\le-\log F^{\#}.$$

As $u\le u_{\rm usc}$ for any function u and $U_k\searrow\log\overline{\rho}$, we therefore have

$$\log\overline{\rho}=\lim_{k\to\infty}U_k\le\lim_{k\to\infty}(U_k)_{\rm usc}\le-\log F^{\#},$$

and the first half of Theorem B follows.

To prove the other half, consider an arbitrary holomorphic function f on Ω which does not vanish identically. Then

$$e_\alpha(x)\ge\frac{|f(x)|^2}{\|f\|_\alpha^2}. \tag{4}$$

Indeed, for $f\in A^2_\alpha$, this is just the definition of $e_\alpha(x)$, and for $f\notin A^2_\alpha$, the right-hand side is zero by the usual convention $1/+\infty=0$. Taking in particular $f=e^{\alpha g}$, we see that

$$e_\alpha(x) \geq \frac{|e^{\alpha g(x)}|^2}{\|e^{\alpha g}\|_\alpha^2} = \frac{|e^{g(x)}|^{2\alpha}}{\int \big(|e^g|^2 F\big)^\alpha d\mu}$$

for any holomorphic function g. Thus

$$e_\alpha(x)^{1/\alpha} \geq \frac{|e^{g(x)}|^2}{\big\||e^g|^2 F\big\|_{L^\alpha(d\mu)}}.$$

Taking the limit gives

$$\underline{\rho}(x) \geq \frac{|e^{g(x)}|^2}{\big\||e^g|^2 F\big\|_{*,d\mu}} \tag{5}$$

where $\|\cdot\|_{*,d\mu}$ is defined as

$$\|\phi\|_{*,d\mu} \equiv \lim_{p\to\infty} \|\phi\|_{L^p(d\mu)} = \begin{cases} \|\phi\|_\infty \text{ if } \phi \in L^p(d\mu)\ \forall p \in (p_0,\infty) \text{ for some finite } p_0, \\ +\infty \text{ otherwise.} \end{cases}$$

Assume further that $|e^g|^2 \leq 1/F$ and $e^{\alpha g} \in A^2_\alpha$ for some $\alpha > 0$. In other words, $|e^g|^2 F \in L^\infty(d\mu) \cap L^\alpha(d\mu)$; thus, since $\log\|\cdot\|_p$ is a convex function of $\frac{1}{p}$, we have $|e^g|^2 F \in L^p(d\mu)\ \forall p \in [\alpha,\infty]$, and $\||e^g|^2 F\|_{*,d\mu} = \||e^g|^2 F\|_\infty$. Consequently,

$$\underline{\rho}(x) \geq \frac{|e^{g(x)}|^2}{\big\||e^g|^2 F\big\|_\infty} \geq |e^{g(x)}|^2.$$

Summing up, we see that

$$\underline{\rho}(x) \geq \sup\{|e^{g(x)}|^2 :\ g \text{ holomorphic on } \Omega, |e^g|^2 \leq 1/F, e^{\alpha g} \in A^2_\alpha \text{ for some } \alpha > 0\},$$

or $\underline{\rho}(x) \geq 1/F^*(x)$, as asserted.

If $\mathbf{1} \in A^2_{\alpha_0}$ for some α_0, we instead take $f = e^{(\alpha-\alpha_0)g}$ in (4). Proceeding as above, we see that

$$e_\alpha(x)^{1/\alpha} \geq \frac{|e^{(\alpha-\alpha_0)g(x)}|^{2/\alpha}}{\big\||e^g|^2 F\big\|_{L^{\alpha-\alpha_0}(d\mu_{\alpha_0})}}$$

and

$$\underline{\rho}(x) \geq \frac{|e^{g(x)}|^2}{\big\||e^g|^2 F\big\|_{*,d\mu_{\alpha_0}}}.$$

Assume that $|e^g|^2 \leq 1/F$. Then $|e^g|^2 F \in L^\infty(d\mu_{\alpha_0})$; owing to the finiteness of $d\mu_{\alpha_0}$, this implies $|e^g|^2 F \in L^p(d\mu_{\alpha_0})\ \forall p > 0$. Thus again $\|\cdot\|_{*,d\mu_{\alpha_0}} = \|\cdot\|_\infty$, and

$$\underline{\rho}(x) \geq \frac{|e^{g(x)}|^2}{\big\||e^g|^2 F\big\|_\infty} \geq |e^{g(x)}|^2,$$

so

$$\underline{\rho}(x) \geq \sup\{|e^{g(x)}|^2 :\ g \text{ holomorphic on } \Omega, |e^g|^2 \leq 1/F\} = 1/F^{**},$$

which is what we wanted to prove. ∎

Proof of Theorem A. Any convex function on $\mathbb{R}^{2n}$ is the supremum of the affine functions lying below it. (An affine function is a sum of a real-linear function and a constant.) Thus, if $-\log F$ is convex, we have

$$\begin{aligned}-\log F &= \sup\{\phi : \ \phi \text{ affine}, \phi \leq -\log F\} \\ &= \sup\{2\operatorname{Re} g : \ g(z) = \langle z, c\rangle + d \ (c \in \mathbb{C}^n, d \in \mathbb{C}), 2\operatorname{Re} g \leq -\log F\}.\end{aligned}$$

Therefore

$$\begin{aligned}1/F &= \sup\{|e^g|^2 : \ g(z) = \langle z, c\rangle + d \ (c \in \mathbb{C}^n, d \in \mathbb{C}), |e^g|^2 \leq 1/F\} \\ &\leq \sup\{|e^g|^2 : \ g \text{ holomorphic on } \Omega, |e^g|^2 \leq 1/F\} = 1/F^{**},\end{aligned}$$

so $F^{**} = F$, and an application of Theorem B completes the proof. $\square$

Remark. The assumption that $\alpha > 0$ in Theorem A can in fact be relaxed to $\alpha \geq 0$. If $\mathbf{1} \in A^2_0$, i.e. if the measure μ is finite, then — as was already noted in the Introduction — we have $F^* = F^{**}$. On the other hand, the preceding paragraph shows that $F^{**} = F$. It only remains to apply Theorem B.

Proof of Theorem B′. Let f be a holomorphic function on Ω, not identically zero, such that $|f|^{2/\gamma} \leq 1/F$ for some $\gamma > 0$. Let us take in (4) $f = f^k$ and $\alpha = \alpha_0 + k\gamma$, where k is an arbitrary positive integer. We obtain

$$e_{\alpha_0+k\gamma}(x) \geq \frac{|f^k(x)|^2}{\int_\Omega |f|^{2k} F^{k\gamma} d\mu_{\alpha_0}}.$$

Therefore

$$e_{\alpha_0+k\gamma}(x)^{1/k\gamma} \geq \frac{|f(x)|^{2/\gamma}}{\big\| |f|^{2/\gamma} F \big\|_{L^{k\gamma}(d\mu_{\alpha_0})}}.$$

Passing to the limit superior as $k \to \infty$, we get

$$\overline{\rho}(x) \geq \limsup_{k\to\infty} e_{\alpha_0+k\gamma}(x)^{1/k\gamma} \geq \frac{|f(x)|^{2/\gamma}}{\big\| |f|^{2/\gamma} F \big\|_{*, d\mu_{\alpha_0}}}.$$

Again, the finiteness of $d\mu_{\alpha_0}$ implies that

$$\big\| |f|^{2/\gamma} F \big\|_{*, d\mu_{\alpha_0}} = \big\| |f|^{2/\gamma} F \big\|_\infty \leq 1,$$

and we conclude that

$$\overline{\rho}(x) \geq |f(x)|^{2/\gamma}$$

for all holomorphic functions f and $\gamma > 0$ such that $|f|^{2/\gamma} \leq 1/F$. By definition (replacing $2/\gamma$ by $\varkappa$), this means that $\overline{\rho} \geq 1/F^{***}$, which completes the proof. $\square$

It would be of interest to know in general for which functions F and G one has $F^{**} = F$, or $F^* = F$. A closely related question is that of characterizing the functions ϕ of the form

$$\phi = \sup\{\psi : \ \psi \leq \phi, \psi \text{ harmonic}\},$$

i.e. the suprema of harmonic functions; the class of all functions of this form which are locally bounded from above is sometimes denoted $\mathcal{H}^{\sup}(\Omega)$ in the literature. Clearly on a simply connected planar domain, $F = F^{**}$ is equivalent to $-\log F \in \mathcal{H}^{\sup}(\Omega)$. Also, any upper semicontinuous function in $\mathcal{H}^{\sup}$ is necessarily plurisubharmonic, since the upper-semicontinuous regularization of a supremum of pluri(sub)harmonic functions is again a plurisubharmonic function. The converse is false: if ϕ is defined on the unit disc as

$$\phi(z) = \max(A, \log|z|)$$

for some constant $A < 0$, then ϕ is subharmonic and any harmonic function $\psi \le \phi$ must satisfy

$$\psi(z) \le A\frac{1-|z|}{1+|z|}$$

by the Harnack inequality; however, the right-hand side is $< \phi(z)$ as soon as $|A|$ is sufficiently large. (The author is indebted to Ivan Netuka [21] for this counterexample.) The class $\mathcal{H}^{\sup}$ has recently been studied by Vondracek [27], [28].

More generally, let $\mathcal{H}(\Omega)$ and $PSH(\Omega)$ stand for harmonic and plurisubharmonic functions on Ω, respectively; denote

$$\begin{aligned} \mathcal{H}_0(\Omega) &= \{\operatorname{Re} f : \ f \text{ holomorphic on } \Omega\}, \\ \mathcal{G}(\Omega) &= \{\varkappa \log|f| : \ f \text{ holomorphic on } \Omega, \varkappa > 0\}, \\ LBA(\Omega) &= \text{ functions on } \Omega \text{ which are locally bounded from above,} \end{aligned}$$

and, in addition to

$$\mathcal{H}^{\sup}(\Omega) = \{\phi \in LBA : \ \phi = \sup_\alpha \psi_\alpha, \psi_\alpha \in \mathcal{H}\}$$

(the suprema of harmonic functions) defined above, introduce the function classes

$$\begin{aligned} \mathcal{H}_0^{\sup} &= \{\phi \in LBA : \ \phi = \sup\nolimits_\alpha \psi_\alpha, \psi_\alpha \in \mathcal{H}_0\} \qquad (\text{suprema of functions from } \mathcal{H}_0), \\ \mathcal{G}^{\sup} &= \{\phi \in LBA : \ \phi = \sup\nolimits_\alpha \psi_\alpha, \psi_\alpha \in \mathcal{G}\} \qquad (\text{suprema of functions from } \mathcal{G}), \\ \mathcal{G}^{\sup}_{\searrow} &= \{\phi : \ \exists \psi_n \in \mathcal{G}^{\sup}, \psi_n \searrow \phi\} \qquad (\text{decreasing limits of function from } \mathcal{G}^{\sup}), \end{aligned}$$

and for a function ϕ locally bounded from above on Ω, define

$$\begin{aligned} \phi_0^{\mathcal{H}} &= \sup\{\psi : \ \psi \in \mathcal{H}_0, \psi \le \phi\} \quad (= \sup\{\psi : \ \psi \in \mathcal{H}_0^{\sup}, \psi \le \phi\}), \\ \phi^{\mathcal{G}} &= \sup\{\psi : \ \psi \in \mathcal{G}, \psi \le \phi\} \quad (= \sup\{\psi : \ \psi \in \mathcal{G}^{\sup}, \psi \le \phi\}), \\ \phi^{\mathcal{G}}_{\searrow} &= \sup\{\psi : \ \psi \in \mathcal{G}^{\sup}_{\searrow}, \psi \le \phi_{\text{usc}}\}, \\ \phi^{PSH} &= \sup\{\psi : \ \psi \in PSH, \psi \le \phi_{\text{usc}}\}. \end{aligned} \tag{6}$$

In particular, for $\phi = -\log F$ these definitions turn into

$$\phi_0^{\mathcal{H}} = -\log F^{**}, \quad \phi^{\mathcal{G}} = -\log F^{***}, \quad \phi^{PSH} = -\log F^{\#}.$$

Clearly we have the containments

$$\mathcal{H}_0^{\sup} \subsetneqq \mathcal{G}^{\sup} \subsetneqq \mathcal{G}^{\sup}_{\searrow}. \tag{7}$$

The first inclusion is immediate, and is strict because of the example in the preceding paragraph. The second inclusion is strict because for $\Omega = \mathbb{D} \setminus \{0\}$, the function

$$\phi(x) = \sum_{j=2}^{\infty} 2^{-j} \log\left|\frac{x - 1/j}{1 - x/j}\right|^2 \tag{8}$$

belongs to $\mathcal{G}^{\sup}_{\searrow}$ (the partial sums of the series on the right-hand side belong to $\mathcal{G}$ and decrease to ϕ), yet any holomorphic function on $\mathbb{D} \setminus \{0\}$ satisfying $|f|^\varkappa \le e^\phi$ with $\varkappa > 0$ is bounded (by 1) and vanishes at $x = \frac{1}{j}$ $(j = 2, 3, \dots)$, hence must be identically 0 by Riemann's Removable Singularities Theorem.

Gathering up the information from our theorems and combining it with (7), we see that our findings so far can be summarized as

$$\phi_0^{\mathcal{H}} \le \phi^{\mathcal{G}} \le \log \overline{\rho} \le \phi^{\mathcal{G}}_{\searrow} \tag{9}$$

and

$$\phi_0^{\mathcal{H}} \le \log \underline{\rho} \le \log \overline{\rho} \le \phi^{PSH} \le \phi_{\text{usc}}$$

where $\phi := -\log F$ and we assume that $\mathbf{1} \in A^2_{\alpha_0}$ for some $\alpha_0 > 0$. Here the third inequality in (9) is a consequence of $\log \overline{\rho} \in \mathcal{G}^{\text{sup}}_{\searrow}$, which in turn follows from (3); the second inequality is the content of Theorem B′. Also, as observed above,

$$F = F^{**} \iff \phi \in \mathcal{H}_0^{\text{sup}} \iff \phi = \phi_0^{\mathcal{H}},$$
$$F = F^{***} \iff \phi \in \mathcal{G}^{\text{sup}} \iff \phi = \phi^{\mathcal{G}},$$
$$F = F^{\#} \iff \phi \in PSH \iff \phi = \phi^{PSH},$$

etc. It would be particularly interesting to know when one has $\phi^{\mathcal{G}} = \phi^{\mathcal{G}}_{\searrow}$, or at least $\phi^{\mathcal{G}} = \phi^{PSH}$. Note that, even though no investigations of the specific situation encountered here are known to the author, the study of various "envelopes" of the form (6) is a standard topic in the literature, in particular in the context of abstract (=Choquet, Shilov, etc.) boundaries; see e.g. the excellent paper on Korovkin theorems by Bauer [2].

3. The limit of $K_\alpha(x,y)^{1/\alpha}$

Proof of Theorem C. By the reproducing property of K_α and the Schwarz inequality, we have

$$|K_\alpha(x,y)|^2 \le K_\alpha(x,x) \cdot K_\alpha(y,y) \equiv e_\alpha(x)e_\alpha(y),$$

so

$$|K_\alpha(x,y)|^{1/\alpha} \le \sqrt{e_\alpha(x)^{1/\alpha} e_\alpha(y)^{1/\alpha}}. \tag{10}$$

Owing to (2) and the hypothesis of local boundedness of $1/F$, it follows that $|K_\alpha(x,y)|^{1/\alpha}$ is locally bounded on $\Omega \times \Omega$, and uniformly so when α ranges through $[1, +\infty)$.

Now let $\alpha_1 < \alpha_2 < \dots$ be a sequence of numbers from A which tend to infinity. Since $K_{\alpha_j}(x,y) \ne 0$ on $U \times U$, it follows from the simple connectivity of U that there exists a single-valued holomorphic branch of $\log K_{\alpha_j}(x, \overline{y})$, $x, \overline{y} \in U$; we can choose this branch to be real on the diagonal $x = \overline{y}$. Define $f_j = K_{\alpha_j}^{1/\alpha_j} = \exp(\frac{1}{\alpha_j} \log K_{\alpha_j})$. By the preceding observation, $f_j(x,y)$ is a locally uniformly bounded family of sesqui-holomorphic (i.e. holomorphic in x and anti-holomorphic in y) functions on $U \times U$. A standard normal family argument shows that there exists a subsequence f_{j_k} which converges to a sesqui-holomorphic function f uniformly on compact subsets of $U \times U$. For $x = y$, we must have $f(x,x) = \rho(x) = 1/F(x)$ by hypothesis. Since each f_j is zero-free, it follows from the Hurwitz theorem ([24], Theorem 3.4.5) — which is easily adapted to the case of several complex variables — that f is either zero-free or identically zero; the latter possibility is, however, ruled out since $1/F \ne \mathbf{0}$. Finally, setting $\alpha = \alpha_{j_k}$ and taking the limit as $k \to \infty$, we see from (10) that

$$|f(x,y)|^2 \le f(x,x)f(y,y).$$

Thus, the function $F(x,y)=1/f(x,y)$ has all the properties required by the theorem. □

4. Some examples

Example 1. Let F,G be such that $A^2_\alpha=\{0\}$ for all α; e.g. $\Omega=\mathbb{C}$, $G=\mathbf{1}$, $F=\mathbf{1}$. Then $e_\alpha(x)=\mathbf{0}$, hence $\lim_{\alpha\to+\infty}e_\alpha(x)^{1/\alpha}\equiv\rho=\mathbf{0}$. This trivial example shows that some additional hypothesis is required to ensure that $\rho(x)=1/F(x)$. Moreover, $F^{**}=F^{***}=F^{\#}=F=\mathbf{1}$ in this case, so we also see that the hypothesis that $\mathbf{1}\in A^2_\alpha$ for some $\alpha>0$ in Theorem B cannot be omitted.

In the remaining examples (except the very last one), we consider the case when Ω is the unit disc $\mathbb{D}$ or the complex plane $\mathbb{C}$, and $F(z)$ and $G(z)$ are *radial* functions, i.e. functions depending only on the modulus $|z|$:

$$F(z)=\Phi(|z|^2),$$
$$G(z)=\gamma(|z|^2).$$

It is then easily verified by passing to polar coordinates (cf. [23], Theorem 0.8, or [11], Proposition 3.11) that

$$\|f\|^2_\alpha=\sum_{n=0}^\infty|f_n|^2\cdot\Big(\pi\int_0^B t^n\Phi(t)^\alpha\gamma(t)\,dt\Big),$$

where f_n are the Taylor coefficients of f and $B=1$ or $+\infty$ for $\Omega=\mathbb{D}$ and $\mathbb{C}$, respectively; moreover, the reproducing kernels are given by

$$K_\alpha(x,y)=\sum_{n=0}^\infty(x\overline{y})^n\Big/\Big(\pi\int_0^B t^n\Phi^\alpha\gamma\,dt\Big), \tag{11}$$

with the convention that $1/+\infty=0$.

Note that the last series converges for

$$|x\overline{y}|<\sup\{t:\ t\in\operatorname{support}(\Phi^\alpha\gamma\,dt)\}.$$

Indeed, the radius of convergence for a series $\sum_0^\infty z_n/c_n$ is equal to $\liminf c_n^{1/n}$, and by the familiar result from abstract measure theory (already alluded to in Section 2), valid for any measure space,

$$\lim_{n\to\infty}\|h\|_n=\begin{cases}\|h\|_\infty & \text{if }\exists\text{ finite }p_0:\ h\in L^p\ \forall p\in(p_0,\infty),\\ +\infty & \text{otherwise,}\end{cases}$$

where $\|h\|_p$ is the L_p norm of a function h.

Example 2. $\Omega=\mathbb{D}$, $G=\mathbf{1}$; $F(z)=\Phi(|z|^2)$, where Φ is continuous on $[0,1]$, $0<\Phi\le\Phi(1)=1$. By the result of the preceding section,

$$\limsup_{\alpha\to+\infty}e_\alpha(x)^{1/\alpha}\equiv\overline{\rho}(x)\le 1/\Phi(|x|^2).$$

It is immediate from (11) that $e_\alpha(x)=K_\alpha(x,x)$ is a non-decreasing function of $|x|^2$. Hence, the same is true for $e_\alpha(x)^{1/\alpha}$ and for the limit $\overline{\rho}(x)$. Thus

$$\overline{\rho}(x)\le\lim_{|x|\to1}\overline{\rho}(x)\le\lim_{|x|\to1}\Phi(|x|^2)^{-1}=1.$$

On the other hand, $\Phi \le 1$ implies that

$$\int_0^1 t^n \Phi^\alpha \, dt \le \frac{1}{n+1}$$

and

$$e_\alpha(x) \ge \pi^{-1}(1-|x|^2)^{-2},$$

so $\liminf_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} \ge 1$. Thus the limit $\rho(x)$ exists and

$$\rho(x) = \lim_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} = 1,$$

regardless of the choice of Φ.

This shows that the map $F \mapsto \rho$ (defined for those F for which the limit ρ exists) is not injective.

Note that $F^* = F^{**} = \mathbf{1}$ by the maximum principle. So in this case, $\rho(x)$ exists and is equal to $1/F^* < 1/F$:

$$1/F^* = 1/F^{**} = 1/F^{***} = \rho = 1/F^\# < 1/F.$$

In general, for any Φ continuous on $[0,1]$ it follows from (11) that $\rho(x)$, if it exists, must be a non-decreasing function of $|x|$; thus a necessary condition for $\rho = 1/F$ is that Φ be non-increasing. As we shall shortly see, even this condition is far from sufficient; still, observe that it implies that (granted $1 \in A^2_\alpha$ for some α)

$$\lim_{\alpha\to+\infty} K_\alpha(0,0)^{1/\alpha} = 1/\lim_{\alpha\to+\infty} \|\Phi\|_{L^\alpha(\gamma\,dt)} = 1/\|\Phi\|_\infty = 1/\Phi(0),$$

i.e. one has at least $\rho(0) = 1/F(0)$. If Φ is C^∞ on $[0,1)$ and has a *strict* maximum at the origin, much more precise information about the asymptotic behaviour of $K_\alpha(0,0)^{1/\alpha}$ can be extracted from (11) by means of the familiar Laplace method (see e.g. [14], § II.1).

Example 3. $\Omega = \mathbb{C}$, $G = \mathbf{1}$, $F(z) = \Phi(|z|^2)$, where

$$\Phi(t) = \begin{cases} A, & 0 \le t \le 1 + 1/A, \\ \dfrac{1}{t-1}, & t \ge 1 + 1/A, \end{cases}$$

A being a positive constant. The integrals in (11) are equal to

$$\int_0^{+\infty} t^n \Phi^\alpha \, dt \equiv c_n = \int_0^{1+1/A} + \int_{1+1/A}^{+\infty} \equiv J_{n,\alpha} + I_{n,\alpha}.$$

Computation gives

$$J_{n,\alpha} = A^\alpha \cdot \left(1 + \frac{1}{A}\right)^{n+1} \cdot \frac{1}{n+1}$$

and

$$I_{0,\alpha} = \frac{A^{\alpha-1}}{\alpha-1} \quad (\alpha > 1), \qquad I_{n,\alpha} = I_{n-1,\alpha} + I_{n-1,\alpha-1},$$

from which it follows that

$$I_{n,\alpha} = \sum_{j=0}^n \binom{n}{j} \frac{A^{\alpha-1-j}}{\alpha-1-j}, \qquad \text{if } 0 \le n \le \alpha - 1,$$

and $I_{n,\alpha}=+\infty$ otherwise. Now on the one hand

$$c_n \ge I_{n,\alpha} \ge \frac{1}{\alpha-1}\sum_{j=0}^{n}\binom{n}{j}A^{\alpha-1-j} = \frac{A^{\alpha-1}}{\alpha-1}\left(1+\frac{1}{A}\right)^n$$

and, for any $t\ge 0$,

$$\begin{aligned}\sum_{0\le n<\alpha-1} t^n/c_n &\le \frac{\alpha-1}{A^{\alpha-1}}\sum_{0\le n<\alpha-1}\left(\frac{t}{1+\frac{1}{A}}\right)^n\\ &\le \frac{(\alpha-1)\alpha}{A^{\alpha-1}}\cdot\left[\max\left(1,\frac{t}{1+\frac{1}{A}}\right)\right]^{\alpha-1}.\end{aligned}$$

It follows that

$$\limsup_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} \le \frac{1}{A}\,\max\left(1,\frac{|x|^2}{1+\frac{1}{A}}\right).$$

On the other hand, for $0\le n<\alpha-2$ we have

$$\begin{aligned}c_n &\le \sum_{j=0}^{n}\binom{n}{j}\frac{A^{\alpha-1-j}}{\alpha-1-n}+J_{n,\alpha}\\ &= A^{\alpha-1}\left(1+\frac{1}{A}\right)^n\cdot\left[\frac{1}{\alpha-1-n}+\frac{A+1}{n+1}\right]\\ &\le A^{\alpha-1}\left(1+\frac{1}{A}\right)^n\cdot(A+2)\end{aligned}$$

and, for any $t\ge 0$,

$$\begin{aligned}\sum_{0\le n<\alpha-1} t^n/c_n &\ge \sum_{0\le n<\alpha-2} t^n/c_n\\ &\ge \frac{A^{1-\alpha}}{A+2}\sum_{0\le n<\alpha-2}\left(\frac{t}{1+\frac{1}{A}}\right)^n\\ &\ge \frac{A^{1-\alpha}}{A+2}\cdot\left[\max\left(1,\frac{t}{1+\frac{1}{A}}\right)\right]^{\alpha-3}\end{aligned}$$

for $\alpha\ge 3$. Consequently,

$$\liminf_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} \ge \frac{1}{A}\,\max\left(1,\frac{|x|^2}{1+\frac{1}{A}}\right).$$

Thus we conclude that

$$\rho(x)\equiv\lim_{\alpha\to+\infty} e_\alpha(x)^{1/\alpha} = \frac{1}{A}\,\max\left(1,\frac{|x|^2}{1+\frac{1}{A}}\right) = \begin{cases}1/A & \text{if } 0\le|x|^2\le 1+1/A,\\ \dfrac{|x|^2}{A+1} & \text{if } |x|^2\ge 1+1/A,\end{cases}$$

and we see that $\rho(x)=1/F(x)$ for $0\le|x|^2\le 1+1/A$, but $\rho(x)<1/F(x)$ for $|x|^2>1+1/A$.

It can be shown that $F^*=F^{**}\equiv A$ in this case. (Use the Borel-Carathéodory theorem (see § 5.5 in [24]), or just plain Cauchy estimates.) Also, taking $\varkappa=2$ and $f(z)=z/\sqrt{A+1}$ shows that $F^{***}=1/\rho$.

Note also that the function $-\log F$ is not convex (so this example does not contradict Theorem A). In fact, it is not even subharmonic, and we finish by showing that its greatest subharmonic minorant $-\log F^{\#}$ is also equal to $\log\rho$, so that we have

$$\mathbf{1} = F^* = F^{**} \gneqq F^{***} = \frac{1}{\rho} = F^{\#} \gneqq F.$$

We already know that $\log\rho$ is subharmonic, so assume that ψ is a subharmonic function satisfying $\log\rho \le \psi \le \phi := -\log F$. Then $\psi \equiv -\log A$ on the disc $|z|^2 \le R := 1 + 1/A$, so it suffices to deal with the region $|z|^2 > R$. Let

$$\chi(x) := \log|x|^2 + \psi(\sqrt{R}/x), \qquad x \in \mathbb{D} \setminus \{0\}.$$

Since the inversion $x \mapsto \sqrt{R}/x$ preserves (sub)harmonicity and $\log|x|^2$ is harmonic on the punctured disc, we see that χ is a subharmonic function on $\mathbb{D} \setminus \{0\}$ which satisfies

$$\log|x|^2 + \log\rho(\sqrt{R}/x) = \log(R-1) \le \chi(x) \le \log|x|^2 + \phi(\sqrt{R}/x) = \log(R - |x|^2).$$

A standard maximum principle argument implies that

$$0 \le \chi(x) - \log(R-1) \le \frac{\log|x|^2}{\log\epsilon} \log\frac{R-\epsilon}{R-1}$$

on the annulus $\epsilon \le |x|^2 \le 1$, for any $1 > \epsilon > 0$. Thus $\chi \equiv \log(R-1)$, or $\psi(\frac{\sqrt{R}}{x}) = \log\frac{R-1}{|x|^2} = \log\rho(\frac{\sqrt{R}}{x})$, so $\psi = \log\rho$ and the assertion follows.

Example 4. $\Omega = \mathbb{C}$, $G = \mathbf{1}$, $F(x) = \Phi(|x|^2)$, where

$$\Phi(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 1/t, & t \ge 1. \end{cases}$$

Proceeding as in the preceding example, we get

$$\int_0^\infty t^n \Phi^\alpha \, dt = \begin{cases} \frac{1}{n+1} - \frac{1}{n-\alpha+1} & \text{if } 0 \le n < \alpha - 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Thus

$$e_\alpha(x) = \frac{1}{\pi} \sum_{0 \le n < \alpha - 1} \frac{(n+1)(\alpha - n - 1)}{\alpha} |x|^{2n}.$$

As before, it is easy to obtain the estimates

$$\pi e_\alpha(x) \le \frac{1}{\alpha} \cdot \alpha^2 \sum_{0 \le n < \alpha-1} |x|^{2n} \le \alpha^2 \big[\max\big(1, |x|^2\big)\big]^\alpha,$$

$$\pi e_\alpha(x) \ge \frac{1}{\alpha} \cdot 1(\alpha - 1) \sum_{0 \le n < \alpha-2} |x|^{2n} \ge \frac{\alpha - 1}{\alpha} \big[\max\big(1, |x|^2\big)\big]^{\alpha-3}.$$

It follows that the limit $\rho(x)$ exists and equals

$$\rho(x) \equiv \lim_{\alpha \to +\infty} e_\alpha(x)^{1/\alpha} = \max(1, |x|^2) = 1/F.$$

On the other hand, if $e^g \equiv f$ satisfies $|f|^2 \le 1/F$, then f is an entire function satisfying

$$|f(z)| \le \max(1, |z|).$$

In view of the Cauchy estimates, this implies that the Taylor coefficients f_n of f vanish for $n > 1$. Thus $f(z) = f_1 z + f_0$, and as $f = e^g$ is necessarily zero-free, we must have $f(z) = f_0 \equiv$const. It follows that $F^{**} = F^* = \mathbf{1} \neq F$. Also, putting $\varkappa = 2$ and $f(z) = z$ in the definition shows that $F^{***} = F$, so, summarizing,

$$\mathbf{1} = F^* = F^{**} \gneqq F^{***} = 1/\rho = F^{\#} = F.$$

This time, we see that $\rho = 1/F$, even though $F^* \neq F$ and $-\log F$ is not convex. Observe, however, that the function $-\log F$ is, at least, subharmonic in this case.

The next two examples are concerned with the convergence of $K_\alpha(x, y)$ on all of $\Omega \times \Omega$ (i.e. not only on the diagonal $x = y$).

Example 5. $\Omega = \mathbb{D}$, $G(x) = 1/|x| = \gamma(|x|^2)$, $F(x) = 1 - |x| = \Phi(|x|^2)$, where $\gamma(t) = 1/\sqrt{t}$, $\Phi(t) = 1 - \sqrt{t}$. We claim that $-\log F$ is a convex function. Indeed, in general, it is well-known that a real-valued, twice continuously differentiable function $f(z)$ defined on a region in the plane is convex if and only if the 2×2 hermitian matrix

$$\begin{pmatrix} \dfrac{\partial^2 f}{\partial z \partial \overline{z}} & \dfrac{\partial^2 f}{\partial z^2} \\ \dfrac{\partial^2 f}{\partial \overline{z}^2} & \dfrac{\partial^2 f}{\partial z \partial \overline{z}} \end{pmatrix}$$

is positive semidefinite. The latter condition can also be written as

$$\frac{\partial^2 f}{\partial z \partial \overline{z}} \geq \left| \frac{\partial^2 f}{\partial z^2} \right|.$$

If, in particular, $f(z) = \phi(|z|^2)$ is a radial function, this reads

$$(t\phi')' \geq |t\phi''|$$

or

$$\phi' \geq 0 \quad \text{and} \quad \phi' + 2t\phi'' \geq 0.$$

In our case $\phi(t) = -\log(1 - \sqrt{t})$, so

$$\phi' = \frac{1}{2\sqrt{t}(1 - \sqrt{t})} > 0, \qquad \phi' + 2t\phi'' = \frac{1}{2(1 - \sqrt{t})^2} > 0,$$

and the claim follows.

By Theorem A, the limit $\rho(x)$ exists and equals $1/F(x)$.

On the other hand, the function $F(x)$ clearly cannot be extended to a function $F(x, y)$ such that $F(x) = F(x, x)$ and $F(x, \overline{y})$ is holomorphic on $\mathbb{D} \times \mathbb{D}$. The only possible candidate is $F(x, y) = 1 - \sqrt{x\overline{y}}$, which is not well defined on $\mathbb{D} \times \mathbb{D}$; however, a single-valued branch exists on $U \times U$ for any simply-connected subregion U of $\mathbb{D}$ not containing the origin. By Theorem C, we conclude that for all sufficiently large α, $K_\alpha(x, y)$ must have a zero at some point and, moreover, these zeroes accumulate at the origin.

In a simple case like this we can verify the last claim directly. Using again the formula (11), a computation shows that (cf. [11], Example 3.31)

$$\begin{aligned} K_\alpha(x, y) &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\Gamma(2n + \alpha + 2)}{\Gamma(2n + 1)\Gamma(\alpha + 1)} (x\overline{y})^n \\ &= \frac{\alpha + 1}{4\pi} \big[(1 - \sqrt{x\overline{y}})^{-\alpha-2} + (1 + \sqrt{x\overline{y}})^{-\alpha-2}\big]. \end{aligned}$$

(Note that this is a single-valued holomorphic function of $x\overline{y}$, even though $\sqrt{x\overline{y}}$ itself is not!) It follows that for any integer k and

$$x\overline{y} = -\tan^2\Big(k+\frac{1}{2}\Big)\frac{\pi}{\alpha} = \tanh^2\frac{(2k+1)\pi i}{2\alpha}$$

we have

$$\frac{1+\sqrt{x\overline{y}}}{1-\sqrt{x\overline{y}}} = e^{\pm(2k+1)\pi i/\alpha}$$

and therefore $K_{\alpha-2}(x,y)=0$.

Example 6. $\Omega=\mathbb{D}$, $G=\mathbf{1}$, and $F(x)=\Phi(|x|^2)$, where Φ is the polynomial

$$\Phi(t) = (t-1)(t+\tfrac{3}{4})(t-\tfrac{11}{4}).$$

The function $\phi=-\log\Phi$ satisfies

$$-\phi' = \frac{\Phi'}{\Phi} = \frac{1}{t-1}+\frac{1}{t+\frac{3}{4}}+\frac{1}{t-\frac{11}{4}} < 0 \qquad \text{on } [0,1],$$

since $\Phi>0$ and $\Phi'<0$ on this interval (cf. [11], Example 3.25); and

$$\phi'' = \frac{1}{(t-1)^2}+\frac{1}{(t+\frac{3}{4})^2}+\frac{1}{(t-\frac{11}{4})^2} > 0,$$

so $\phi'+2t\phi''\geq 0$ on $[0,1]$. As in the preceding example, it follows that $-\log F$ is a convex function, and as Φ is bounded, Theorem A applies. By Theorem C, if there existed a sequence $\alpha_j\to\infty$ such that each $K_{\alpha_j}(x,y)$ were zero-free on $\mathbb{D}\times\mathbb{D}$, then

$$\lim_{j\to\infty} K_{\alpha_j}(x,y)^{1/\alpha_j} = 1/F(x,y),$$

where $F(x,y)$ would be a sesqui-holomorphic extension of $F(x)$ to $\mathbb{D}\times\mathbb{D}$. The only possible such extension is given by (cf. [7], Theorem II.7)

$$F(x,y) = (x\overline{y}-1)(x\overline{y}+\tfrac{3}{4})(x\overline{y}-\tfrac{11}{4}).$$

However, taking $x=-y=\sqrt{3}/2$ gives $F(x,y)=0$, so $F(x,y)$ is not zero-free, and

$$0 = |F(x,y)|^2 < F(x,x)F(y,y) = \frac{9}{16}$$

so the "reverse Schwarz" inequality is likewise violated. It follows that for all sufficiently large α, $K_\alpha(x,y)$ must have a zero.

Example 7. In this final example we exhibit a situation in which $\overline{\rho}\gneqq 1/F^{***}$. (In other words, the assertion of Theorem B$'$ is not the sharpest one possible.) To that end, consider the function (8):

$$\phi(x) = \sum_{j=2}^{\infty} 2^{-j}\log|b_j(x)|^2, \qquad b_j(x) := \frac{x-1/j}{1-x/j}.$$

Clearly ϕ is subharmonic (hence, upper semicontinuous) and

$$(*) \qquad \phi(1/j) = -\infty, \qquad j=2,3,\dots,$$

while, on the other hand,

$$\phi(0) = -2\sum_{j=2}^{\infty}\frac{\log j}{2^j} > -\infty.$$

Pick a number ϵ, $0<\epsilon<e^{\phi(0)}$, and let

$$\phi_\epsilon = \max(\phi, \log\epsilon).$$

Now take $\Omega=\mathbb{D}$, $G=\pi^{-1}\mathbf{1}$ and $F=e^{-\phi_\epsilon}$. Owing to $(*)$, any continuous function Ψ lying below $1/F$ must satisfy $\Psi(1/j)\le\epsilon$, $j=2,3,\dots$, and, hence, also $\Psi(0)\le\epsilon$. It follows that

$$1/F^*(0)=1/F^{**}(0)=1/F^{***}(0)=\epsilon<e^{\phi(0)}=1/F(0)=1/F^{\#}(0).$$

Let us now obtain a bound for $\overline{\rho}(0)$. By (4),

$$e_\alpha(0)^{1/\alpha}\ge\frac{|f(0)|^{2/\alpha}}{\|f\|_\alpha^{2/\alpha}}$$

for any analytic function f which does not vanish identically. Let us take $\alpha=2^n$ and

$$f(x)=\prod_{j=2}^n b_j(x)^{2^{n-j}}.$$

We have $|f(0)|^{2/2^n}=\exp(-2\sum_{j=2}^n 2^{-j}\log j)$, which tends to $e^{\phi(0)}$ if n goes to infinity. On the other hand,

$$\begin{aligned}\|f\|_{2^n}^{2/2^n}&=\Big(\int_{\mathbb{D}}|f|^2\exp(-2^n\phi_\epsilon)\,d\mu\Big)^{1/2^n}\\&=\Bigg(\int_{\mathbb{D}}\frac{\prod_{j=2}^n|b_j(x)|^{2^{n+1-j}}}{\max\Big(\prod_{j=2}^\infty|b_j(x)|^{2^{n+1-j}},\epsilon^{2^n}\Big)}\,d\mu(x)\Bigg)^{1/2^n}\\&=\Bigg(\int_{\mathbb{D}}\min\Bigg(\frac{\prod_{j=2}^n|b_j|^{2^{n+1-j}}}{\epsilon^{2^n}},\frac{1}{\prod_{j=n+1}^\infty|b_j|^{2^{n+1-j}}}\Bigg)\,d\mu\Bigg)^{1/2^n}\equiv\|f_n\|_{L^{2^n}(d\mu)},\end{aligned}$$

where

$$f_n:=\min\Bigg(\frac{\prod_{j=2}^n|b_j|^{2^{1-j}}}{\epsilon},\frac{1}{\prod_{j=n+1}^\infty|b_j|^{2^{1-j}}}\Bigg).$$

Thus we have arrived at

$$\overline{\rho}(0)\ge\frac{e^{\phi(0)}}{\lim\limits_{n\to\infty}\|f_n\|_{L^{2^n}(d\mu)}}.$$

We claim that the limit in the denominator equals one. To see this, observe first of all that $f_{n+1}=|b_{n+1}|^{2^{-n}}f_n$, so by the standard property of the Blaschke products $f_{n+1}\le f_n$. Hence, $f_n\ge f_{n+1}\ge f_{n+2}\ge\dots\ge f_\infty$, where

$$f_\infty:=\lim_{n\to\infty}f_n=\min(e^\phi/\epsilon,1).$$

Owing to the finiteness of $d\mu$, it therefore follows that

$$\|f_n\|_{L^{2^n}(d\mu)}\ge\|f_\infty\|_{L^{2^n}(d\mu)}\to\|f_\infty\|_\infty\ge f_\infty(0)=\min\Big(\frac{e^{\phi(0)}}{\epsilon},1\Big)=1.$$

On the other hand, since the convergence of the sum (8) is locally uniform as long as we stay away from the points j and $1/j$ ($j=2,3,\dots$), the functions ϕ, f_n and f_∞ extend continuously to the boundary of the unit disc, and $f_n=f_\infty=1$ there. By Dini's theorem, $f_n\searrow f_\infty$ therefore implies $\|f_n\|_\infty\to\|f_\infty\|_\infty$, and, further, the fact that $d\mu$ is a probability measure implies that $\|\cdot\|_{L^p(d\mu)}$ is a nondecreasing function of p, by Hölder's inequality; consequently,

$$\|f_n\|_{L^{2^n}(d\mu)} \le \|f_n\|_\infty \to \|f_\infty\|_\infty = \|\min(e^\phi/\epsilon, 1)\|_\infty \le 1.$$

Thus, indeed, $\lim_{n\to\infty} \|f_n\|_{L^{2^n}(d\mu)} = 1$, and

$$\overline{\rho}(0) \ge e^{\phi(0)}.$$

Hence $\overline{\rho}(0) = e^{\phi(0)}$. Summing everything up, we see that in this case

$$1/F^*(0) = 1/F^{**}(0) = 1/F^{***}(0) < \overline{\rho}(0) = 1/F^\#(0) = 1/F(0),$$

as we have asserted.

5. Postscript: A few open problems

(I) The author does not know of any situation in which the limit $\rho(x)$ would fail to exist. Is it true that this limit always exists?

(II) If the answer to (I) is affirmative, is there a neat formula for the limit? For instance, can it be true that

$$\rho = 1/F^\#$$

whenever $\mathbf{1} \in A^2_\alpha$ for some α? Note that this gives the correct answer in all the examples above.

(III) Characterize the functions F for which (a) $F = F^{**}$, or (b) $F = F^{***}$. In other words, give an "easy" criterion for a function $\phi = -\log F$ to belong to $\mathcal{H}_0^{\sup}(\Omega)$ or $\mathcal{G}^{\sup}(\Omega)$, in the notation (6).

(IV) Adding yet another definition to (6), let

$$\mathcal{G}^\infty(\Omega) := \{\sum_{j=1}^\infty \varkappa_j \log|f_j| : \ f_j \text{ holomorphic on } \Omega, \varkappa_j > 0\},$$

and

$$\phi^\infty := \sup\{\psi : \ \psi \in \mathcal{G}^\infty, \psi \le \phi\}.$$

Is it true that $\phi = \phi^\infty$ for any PSH function ϕ? (Observe that if we used only finite sums in the definition of $\mathcal{G}^\infty$, then, by an easy approximation argument, ϕ^∞ would be just the same thing as $\phi^{\mathcal{G}}$.)

It would also be of interest to clarify the relation between $\mathcal{G}^{\sup}_{\searrow}$ and PSH.

(V) In the applications in quantization, Ω is a Kähler manifold whose Kähler metric $ds^2 = g_{i\overline{j}}\, dz^i d\overline{z}^j$ is given by (in local coordinates)

$$g_{i\overline{j}} = \frac{\partial^2 \Psi}{\partial z^i \partial \overline{z}^j},$$

where Ψ is a real-valued function on Ω (the Kähler potential); one then takes $F = e^{-\Psi}$ and $G = \det(g_{i\overline{j}})$ (the volume form). Thus, in view of the positive-definiteness of the metric tensor $g_{i\overline{j}}$, the function $-\log F = \Psi$ is automatically strictly plurisubharmonic. What are the Kähler manifolds for which $F = F^*$? What are the ones for which Ψ belongs to $\mathcal{H}_0^{\sup}(\Omega)$ or $\mathcal{G}^{\sup}(\Omega)$?

(VI) In the applications to quantization, one further needs something stronger than the equality $\rho(x) = 1/F(x)$ or even $\lim K_\alpha(x,y)^{1/\alpha} = 1/F(x,y)$. What is needed is that

$$\lim_{j\to\infty} \frac{K_{\alpha_j}(x,y)F(x,y)^{\alpha_j}}{\alpha_j^{\dim\Omega}} = \mathbf{1} \tag{12}$$

for some sequence α_j tending to infinity. (The numbers $1/\alpha_j$ then correspond to the admissible values of the Planck constant.) This presupposes that $F(x) = F(x,x)$ for some sesqui-holomorphic function $F(x,y)$ on $\Omega \times \Omega$, and that $\rho(x)$ exists and equals $1/F(x)$. It would be desirable to strengthen the results of the present paper so as to obtain (12) instead of (1).

(VII) Observe that the case $G = \mathbf{1}$ (or, upon replacing α by $\alpha - \beta$, which has no influence on the limit $\rho(x)$, $G = F^\beta$ for some real β) corresponds to the case when the metric $g_{i\bar{j}}$ has "constant curvature" — more precisely: when it is a Kähler-Einstein metric. Can some of the problems above be solved at least in this important case? It is known that a complete Kähler-Einstein metric exists e.g. on any bounded pseudoconvex domain in $\mathbb{C}^n$, and is unique (up to rescaling) if the domain is strongly pseudoconvex (see, for instance, the survey article by Wu [29]).

We remark that the completeness of the metric corresponds to the function F having a zero on $\partial\Omega$ of precisely the first order. Thus dealing with complete metrics automatically rules out such pathological situations as in Example 2.

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